Determine whether or not the domain of the given functions is all real numbers.

$$g(x) = \frac{x}{1+x^2}$$

Domain $= \mathbb{R}$

$$f(x) = \frac{x+1}{x^2-1}$$

$x^2-1=0 \Rightarrow x=1, x=-1$

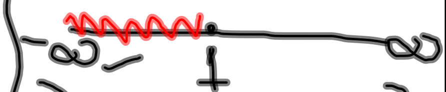
Domain: $\mathbb{R} \setminus \{-1,1\}$

$$h(t) = \sqrt{1-t}$$

≥ 0

$1-t \geq 0$

$1 \geq t$



Domain: $(-\infty, 1]$

Find the domain of $f(x) = \dfrac{x+2}{\sqrt{x^2-9}}$

1. $x^2 - 9 > 0$. $x^2 - 9 = 0 \Rightarrow x = \pm 3$.

x	$-\infty$		-3		3		∞
x^2-9		$+$		$-$		$+$	

Domain: $(-\infty, -3) \cup (3, \infty)$

example. For given f and g, find $(f\circ g)(x)$ and $(g\circ f)(x)$. Find x (if any) such that $(f\circ g)(x) = (g\circ f)(x)$. $f(x)=\sqrt{x}$, $g(x)=1-3x$

$(f\circ g)(x) = f(g(x)) = \sqrt{1-3x}$

$(g\circ f)(x) = g(f(x)) = 1-3\cdot\sqrt{x}$

$(\sqrt{1-3x})^2 = (1-3\sqrt{x})^2 \Rightarrow \cancel{1}-3x = \cancel{1}-6\sqrt{x}+9x \Rightarrow$

$\Rightarrow 6\sqrt{x} = 12x \Rightarrow (\sqrt{x})^2 = (2x)^2 \Rightarrow x = 4x^2,$

$x(1-4x)=0$. Then $x=0$ or $x=1/4$. But x can not be 1/4. Why !!!!!!!!!!!!!!!

$$P(x) = -2.5x^2 + 47.5x - 85 > 0$$

$$P(x) = -2.5(x^2 - 19x + 34)$$

$$= -2.5\times(x-17)\cdot(x-2) > 0$$

$\cancel{x>17, x<2}$ $\leftarrow$ >0 $\quad$ <0

$\leftarrow$ <0 $\quad$ >0

$x<17, x>2$ $\quad \boxed{x \in (2,17)}$ $\quad x = 3, 4, 5, 6, \ldots\ldots 17$

For given f and g, find $(f\circ g)(x)$ and $(g\circ f)(x)$.
Find x (if any) such that $(f\circ g)(x) = (g\circ f)(x)$.

$f(x) = \sqrt{x}$, $g(x) = 1-3x$.

$(f\circ g)(x) = f(g(x)) = \sqrt{1-3x}$ $x=1/4$

$(g\circ f)(x) = g(f(x)) = 1-3\cdot\sqrt{x}$ $x=1/4$

$\sqrt{1-\frac{3}{4}} = 1/2$

$\neq$

$1-\frac{3}{2} = -1/2$

$\left(\sqrt{1-3x}\right)^2 = \left(1-3\sqrt{x}\right)^2$

$\cancel{1}-3x = \cancel{1}-6\sqrt{x}+9x$

$6\sqrt{x} = 12x \Rightarrow \left(\sqrt{x}\right)^2 = (2x)^2$

$x = 4x^2$

$x-4x^2 = 0$

$x(1-4x) = 0$

$0 = x$

$x = 1/4$

$(a)^2 = (2)^2$

$a^2 = 4$

$a=2$, $a=-2$

$(a-b)^2 = a^2-2ab+b^2$

$1-4x=0$

$1=4x$

$x=1/4$

$$P(x) = \underbrace{-2.5x^2}_{-2.5} \underbrace{+47.5x}_{-2.5} \underbrace{-85}_{-2.5} > 0$$

$$= -2.5\left(x^2 - 19x + 34\right)$$

$$= -2.5(x-17)(x-2) > 0$$

Case 1: >0 <0 $\rightarrow \cancel{x>17,\ x<2}$

Case 2: <0 >0 $\rightarrow x<17,\ x>2$

$$\boxed{x \in (2, 17)}$$

$$y(x) = x^3 - x$$

$$y(0) = 0 - 0 = 0$$

$$y(1) = 1^3 - 1 = 0$$

$$y(2) = 2^3 - 2 = 6$$

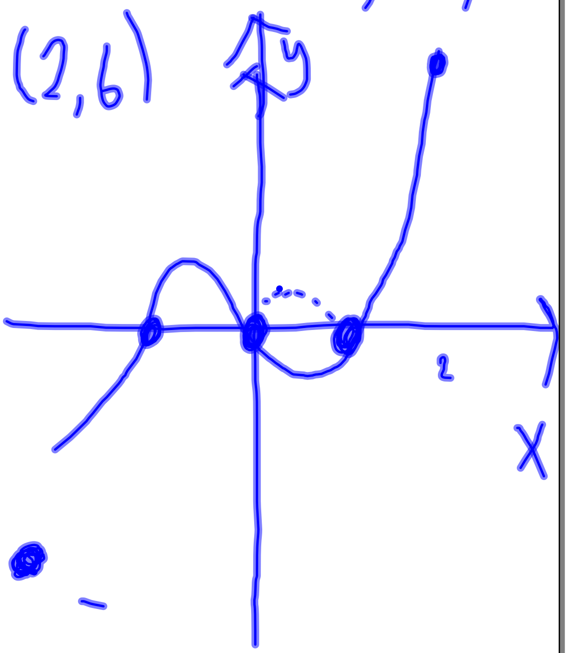
$$y(-1) = -1 - (-1) = 0$$

$$y(-2) = -8 - (-2) = -6$$

$(0,0)$ $(-1,0)$

$(1,0)$ $(-2,-6)$

$(2,6)$



$$y(x) = \sqrt{3-x}$$

$$3-x \geq 0 \Rightarrow 3-x \geq 0$$

Domain:

$$(-\infty, 3] \Rightarrow \boxed{x \leq 3}$$

$y(x) = -x^2 + 12x - 11$

S1) $x = -\frac{b}{2a} = -\frac{12}{-2} = 6$

$x = 6 \quad y = 25$

S2) $A = -1 < 0$, open down

S3) $x = 0 \Rightarrow y = -11$

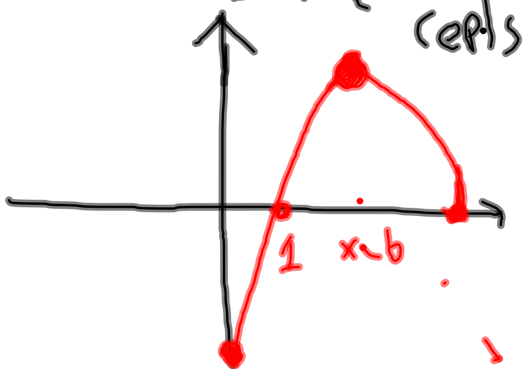
y-intercept

$y = 0 \Rightarrow -x^2 + 12x - 11 = 0$

$x^2 - 12x + 11 = 0$

$(x-1)(x-11) = 0$

$x = 1 \quad x = 11$ (x-intercepts)



Standart Form:

$$y = mx + b$$

Slope Constant Term

$$y = \boxed{\frac{3}{4}} x - 3$$

$$y = \boxed{m} x + b$$

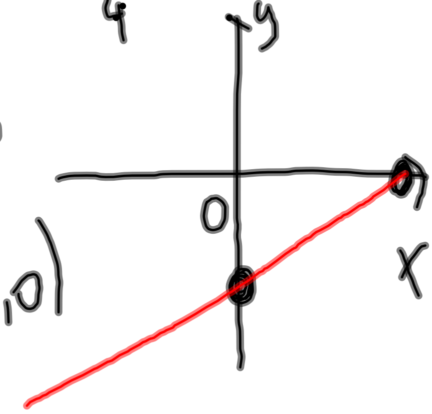
$m = 3/4 =$ Slope

$x = 0 \Rightarrow y = -3 \quad (0, -3)$

$y = 0 \Rightarrow 0 = \frac{3}{4} x - 3$

$\frac{3}{4} x = 3$

$x = 4 \quad (4, 0)$



$(-1, 4)$, $(-5, -8)$

(x_1, y_1) (x_2, y_2)

Slope: $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-8-4}{-5-(-1)} = \dfrac{-12}{-4} = 3$

$y = 3x + b$

Use $(-1, 4)$

$4 = -3 + b$

$b = 7$

Use $(-5, -8)$

$-8 = -15 + b$

$b = 7$

$\boxed{y = 3x + 7}$

Find eq. of the line through $(6,-2)$, $(-4,1)$ by Slope-point Formula.

$$m = \frac{1-(-2)}{-4-6}$$

$$m = \frac{3}{-10} = -3/10$$

Use $(6,-2)$

$x_0 \quad y_0$

$y - y_0 = m(x - x_0)$

$$y+2 = -\frac{3}{10}(x-6)$$

$10y + 20 = -3x + 18$

$10y = -3x - 2$

$$\boxed{y = -\frac{3}{10}x - \frac{1}{5}}$$

Use $(-4,1)$

$x_0 \quad y_0$

$$y - 1 = -\frac{3}{10}(x+4)$$

$10y - 10 = -3x - 12$

$10y = -3x - 2$

$$\boxed{y = -\frac{3}{10}x - \frac{1}{5}}$$

$2y - x - 9 = 0$

Put it into standart Form:

$2y = x + 9$

$y = \boxed{\frac{1}{2}} x + \frac{9}{2}$

$y = 6 - x$

$y = \boxed{-1} x + 6$

$\frac{1}{2} \neq -1$, not parallel

b) $\frac{1}{2} \cdot \text{slope} = -1$ Slope: -2

ℓ: Slope: -2, through $(1, 4)$

Use point-slope formula

$$y - 4 = -2(x - 1)$$

continue

Find all points of instersection of $f(x)=3x+2$, $g(x)=x^2$.

We set $f(x)=g(x)$ and solve for x.

$x^2=3x+2 \Rightarrow x^2-3x-2=0$.

$ax^2+bx+c=0$ | $a=1$, $b=-3$, $c=-2$.

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ | $x=\frac{-(-3)\pm\sqrt{9-(-8)}}{2\cdot 1}$ | $x=\frac{3+\sqrt{17}}{2}$. $x=\frac{3-\sqrt{17}}{2}$

$R(x) = 15x - x^2 , \quad C(x) = 3x + 11$

We find break-even point by solving $R(x) = C(x)$

$$15x - x^2 = 3x + 11 \Rightarrow 0 = x^2 - 12x + 11 = 0$$

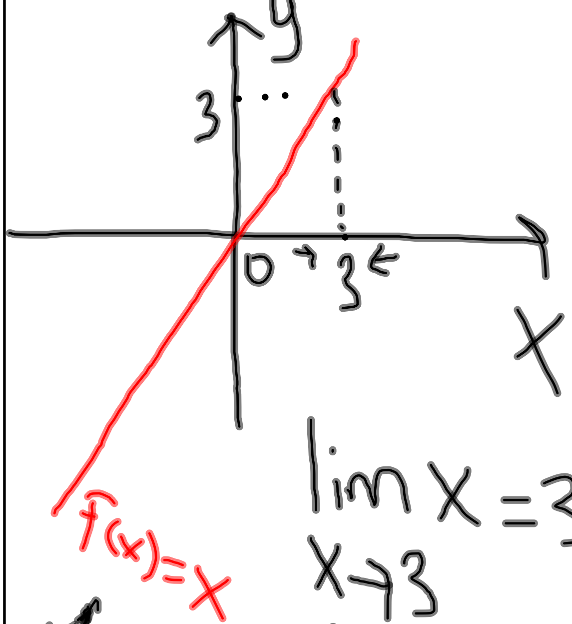
$+x^2$

0

$-15x$

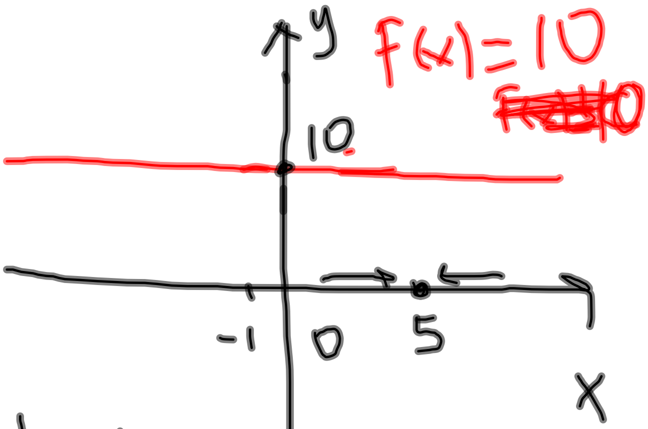
$$(x-11)\cdot(x-1) = 0$$

$$\boxed{x = 11 , \quad x = 1}$$



$$\lim_{x \to 3} x = 3$$

$$\lim_{x \to a} x = a \qquad \lim_{x \to 2} x = 2$$



$$\lim_{x \to 5} 10 = 10 \qquad \lim_{x \to a} c = c$$

$$\lim_{x \to -1} 10 = 10$$

Find $\lim_{x \to 1} (x^4 - x + 1)$

$$= \lim_{x \to 1} x^4 - \lim_{x \to 1} x + \lim_{x \to 1} 1$$

$$= \left[\lim_{x \to 1} x\right]^4 - \lim_{x \to 1} x + \lim_{x \to 1} 1 = 1^4 - 1 + 1$$

$$= 1$$

$$\lim_{x \to 3} (x^3 - 4) = \left[\lim_{x \to 3} x\right]^3 - \lim_{x \to 3} 4$$

$$= 3^3 - 4 = 23$$

Find $\lim\limits_{x \to 1} \boxed{\dfrac{x^3+1}{x^2+2}} = R(x)$

R(1) is defined. So the limit is equal to R(1). That is

$$\lim_{x \to 1} R(x) = R(1) = \frac{1^3+1}{1^2+2} = \frac{2}{3}.$$

$$\lim_{x \to 2} \frac{x^2-4}{\underbrace{x^2+x-6}_{2^2+2-6=0}} = ? \qquad \frac{0}{0} \text{ indetermined}$$

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{(x+3)\cancel{(x-2)}} = \lim_{x \to 2} \frac{x+2}{x+3}$$

$$= 4/5$$

$$\lim_{x \to +\infty} \frac{1}{x^5} = 0$$

$$\lim_{x \to +\infty} \frac{-5}{x^4} = 0$$

#1. Find $\lim_{x \to \infty} \dfrac{x^2}{1+x+2x^2}$.

Method: Divide each term in $f(x)$ by the highest power of x that appears in the denominator.

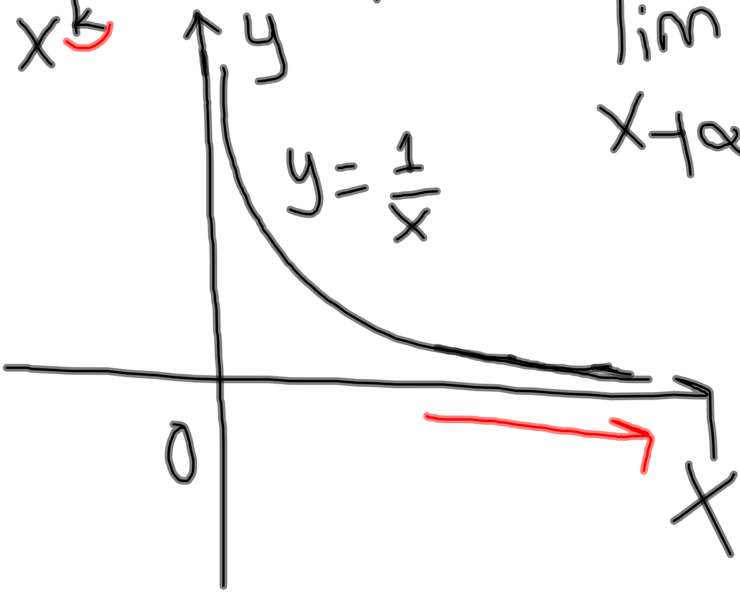
Reminder: $\lim_{x \to \pm\infty} \dfrac{A}{x^k} = 0, \; k > 0$

$$\lim_{x\to+\infty} \frac{x^2}{1+x+2x^2} = ?$$ We divide each term by x^2.

$$= \lim_{x\to+\infty} \frac{x^2/x^2}{\frac{1}{x^2}+\frac{x}{x^2}+\frac{2x^2}{x^2}} = \lim_{x\to+\infty} \frac{1}{\frac{1}{x^2}+\frac{1}{x}+2}$$

$$= \frac{\lim_{x\to+\infty} 1}{\lim_{x\to+\infty}\left(\cancelto{0}{\frac{1}{x^2}}+\cancelto{0}{\frac{1}{x}}+2\right)} = 1/2.$$

$$\lim_{x \to \pm\infty} \frac{A}{x^k} = 0, \quad k > 0$$



$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

Find $\lim\limits_{x \to \infty} \dfrac{2x^2+3x+1}{3x^2-5x+2}$. We divide each term by x^2

$$= \lim_{x \to \infty} \frac{\frac{2x^2}{x^2}+\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}-\frac{5x}{x^2}+\frac{2}{x^2}} = \frac{\lim\limits_{x\to\infty}\left(2+\cancel{\frac{3}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}{\lim\limits_{x\to\infty}\left(3-\cancel{\frac{5}{x}}_{\,0}+\cancel{\frac{2}{x^2}}_{\,0}\right)} =$$

$$= 2/3.$$

#3 Find $\lim_{x \to 1} \dfrac{\sqrt{x}-1}{x-1}$. By putting $x=1$, $\dfrac{\sqrt{1}-1}{1-1} = \dfrac{0}{0}$ (indetermined)

$x-1 = (\sqrt{x})^2 - 1^2 = (\sqrt{x}-1)(\sqrt{x}+1)$

a^2-b^2

$(a-b)(a+b)$

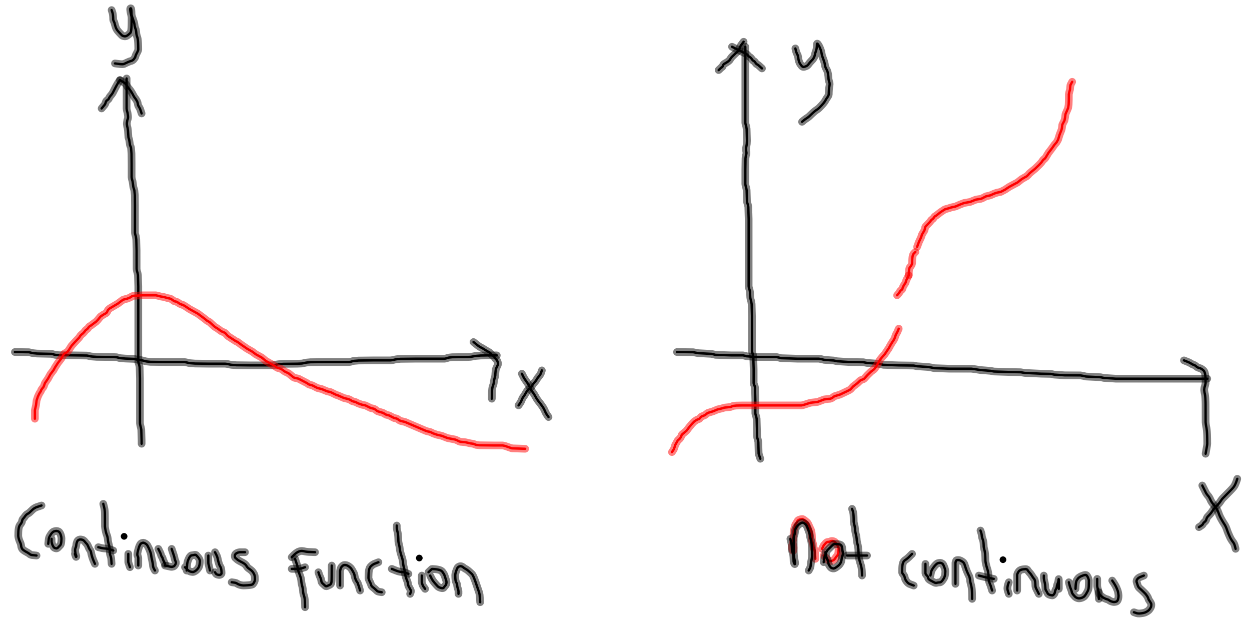
$$\lim_{x \to 1} \frac{\cancel{\sqrt{x}-1}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)} = \frac{1}{2}.$$

$$C(x) = \underbrace{7.5x}_{\text{Variable Cost}} + \underbrace{120.000}_{\text{Fixed Cost}}$$

$$AC(x) = \frac{C(x)}{x} = \frac{7.5x + 120.000}{x}$$

$$\lim_{x \to \infty} AC(x) = \frac{\frac{7.5\cancel{x}}{\cancel{x}} + \cancel{\frac{120.000}{x}}^{\,0}}{x/x} = 7.5$$

As the number of units that produced increases, then the contribution of the fixed cost to the average cost becomes zero.


y
x
Continuous function
y
x
Not continuous

Let $f(x) = \begin{cases} 1 - x^2, & x \leq 1 \\ 3x + 1 & x > 1 \end{cases}$

a) $\lim\limits_{x \to 1^+} f(x) = 3 \cdot 1 + 1 = 4$

b) $\lim\limits_{x \to 1^-} f(x) = 1 - 1^2 = 0$

c) $\lim\limits_{x \to 1} f(x)$

Since $4 \neq 0$, the limit does not exist.

$$\lim_{x\to 0} f(x) = 1 - 0^2 = 1$$

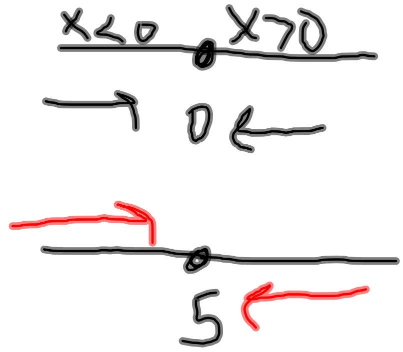
$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

$$\lim_{x\to 5} f(x) = 3\cdot 5 + 1$$

$$= 16$$

$$= \lim_{x\to 5^+} f(x) = \lim_{x\to 5^-} f(x)$$

$$f(x) = \begin{cases} 1 - x^2, & x \le 1 \\ 3x + 1, & x > 1 \end{cases}$$



$\lim_{x \to a} f(x) = f(a)$:

① $f(x)$ is defined at $x=a$. ✓

② $\lim_{x \to a} f(x)$ exists. That is, $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$. ✓

③ limiting value $\lim_{x \to a} f(x)$ = function's value $f(a)$

The function $f(x) = \begin{cases} 1-x^2, & x \leq 1 \\ 3x+1, & x > 1 \end{cases}$

is not continuous at $x = 1$ because $\lim_{x \to 1} f(x)$ does not exist!!!!

$$h(t) = \begin{cases} 5+t, & 0 \le t < 6, \\ \underbrace{b}_{11} - 16(t-6)^2, & t \geq 6 \end{cases}$$

① $h(6) = b$

② $\lim\limits_{t \to 6^-} h(t) = 5+6 = 11 = \lim\limits_{t \to 6^+} h(t) = b - 16(6-6)^2 = b$

We conclude $\boxed{b = 11}$

③ $\lim\limits_{t \to 6} f(t) = f(6)$

$11 = 11$

For functions given below, find the limits:

1. $\lim_{x \to 4^+} (3x^2 - 9) = 39$

$\lim_{x \to 4^-} (3x^2 - 9) = 39$

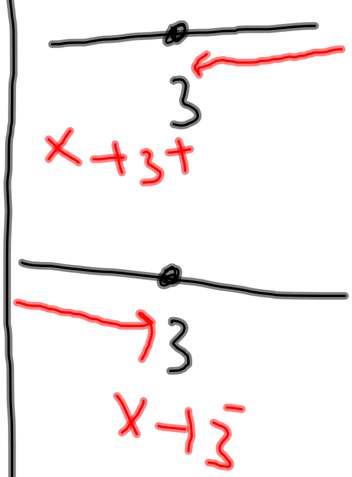
$\lim_{x \to 4} (3x^2 - 9) = 3 \cdot 4^2 - 9 = 39$

Reminder:
$\lim_{x \to a} R(x) = R(a)$,
if $R(a)$ is defined.

② $\lim\limits_{\substack{x\to 3^+ \\ (x>3)}} \sqrt{\underbrace{3x-9}_{>0}} = 0$

$\lim\limits_{\substack{x\to 3^- \\ (x<3)}} \sqrt{\underbrace{3x-9}_{<0}}$ does not exist.

$\lim\limits_{x\to 3} \sqrt{3x-9}$ = does not exist.



(3) $\lim\limits_{x \to 2^+} \dfrac{x^2+4}{x-2} = +\infty$

$(x>2)$

$\lim\limits_{x \to 2^-} \dfrac{x^2+4}{x-2} = -\infty$

$(x<2)$

$\lim\limits_{x \to 2} \dfrac{x^2+4}{x-2} =$ does not exist.

$\lim\limits_{x \to 3} \dfrac{x^2+4}{x-2}$

$= \dfrac{3^2+4}{3-2}$

$= 13.$

(4) $\lim_{x \to 1^+} \frac{x - \sqrt{x}}{x-1} = \infty$

$\lim_{x \to 1^-} \frac{x - \sqrt{x}}{x-1} = -\infty$

$\lim_{x \to 1} \frac{x - \sqrt{x}}{x-1}$ DNE (does not exist)

$\lim_{x \to 2} \frac{x - \sqrt{x}}{x-1}$

$= 2 - \sqrt{2}$.

Examples

(1) Let $f(x) = \begin{cases} x^2 - 4, & x \le 0, \\ 2x - 4, & 0 \le x < 4, \\ -x^2, & x \geq 4. \end{cases}$

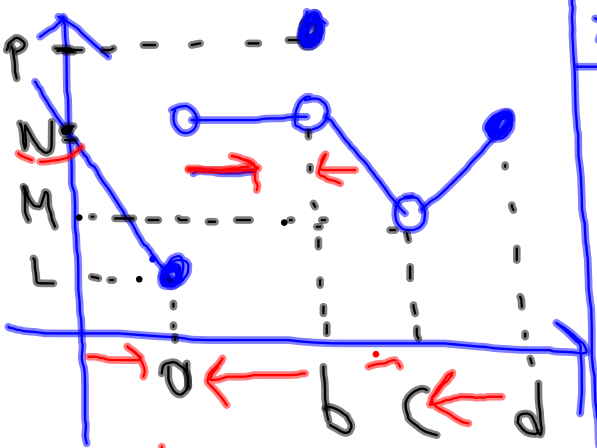
a) $\lim\limits_{x \to 0} f(x)$? $= -4$

$\lim\limits_{x \to 0^+} f(x) = 2 \cdot 0 - 4 = -4$, $\lim\limits_{x \to 0^-} f(x) = 0^2 - 4 = -4$

b) $\lim\limits_{x \to 4} f(x)$? D.N.E. Since $4 \neq -16$.

$\lim\limits_{x \to 4^+} f(x) = -(4^2) = -16$, $\lim\limits_{x \to 4^-} f(x) = 2 \cdot 4 - 4 = 4$

② The graph of f(x) is given below. Find indicated limits.



Since $\lim_{x\to c^-} f(x) = M$

$\lim_{x\to a^+} f(x) = N$ $(x\to a)$

$\lim_{x\to a} f(x) =$ D.N.E (does not exist) Since $\lim_{x\to a^-} f(x) = L \neq N$

$\lim_{x\to b} f(x) = N$

$\lim_{x\to c^+} f(x) = M$

$\lim_{x\to c} f(x) = M$

③ Find $\lim_{x \to 0} \frac{|x|}{x}$. Reminder: $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

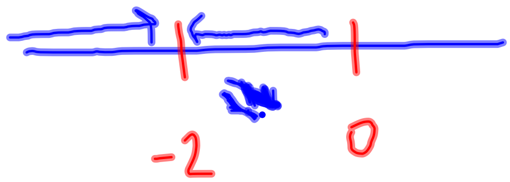
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1.$ $(x > 0)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$ $(x < 0)$

$-x = |x|$

Since $1 \neq -1$, the limit DNE

4) $\lim\limits_{x \to -2} \dfrac{|x|}{x}$.

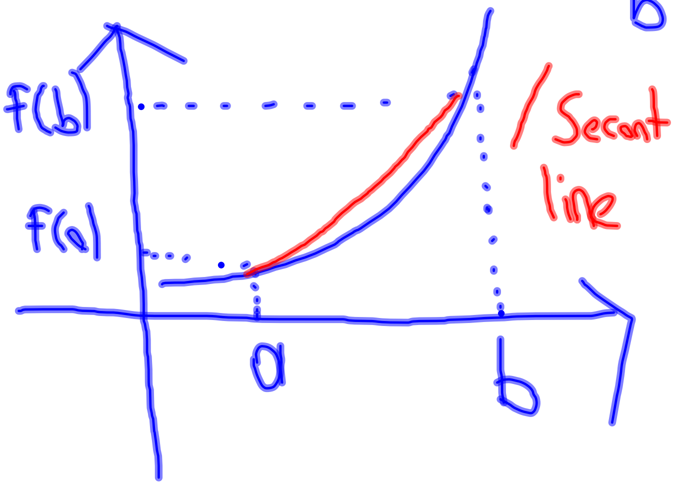


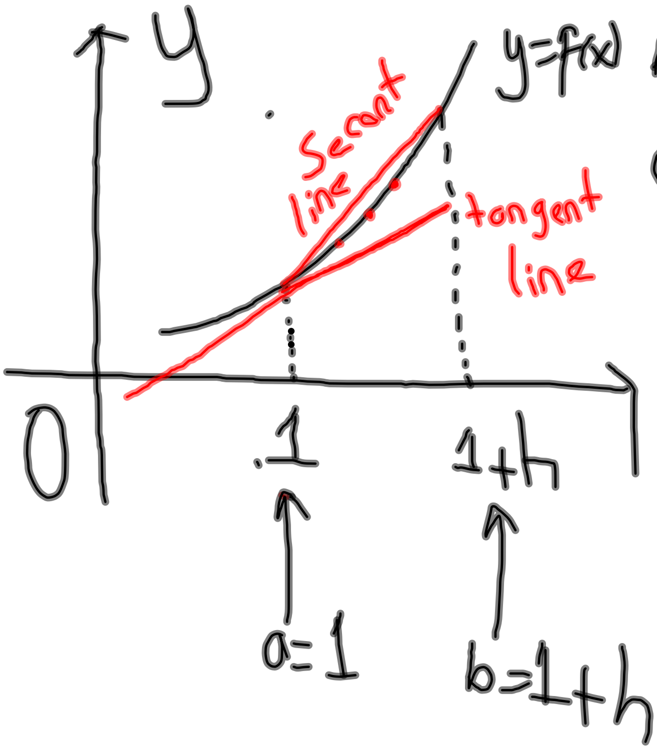
$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \dfrac{-x}{x} = -1$

$\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^+} \dfrac{-x}{x} = -1$

$\lim\limits_{x \to -2} \dfrac{|x|}{x} = -1$.

The average rate of change on $[a,b]$ is $\frac{f(b)-f(a)}{b-a}$.





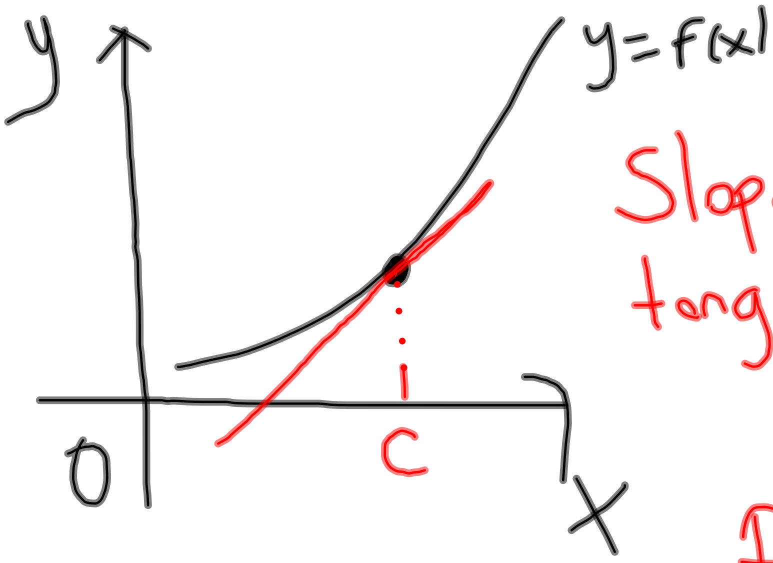
Average rate of Change at $x=1$?

$$\boxed{\lim_{h\to 0}\frac{f(1+h)-f(1)}{\cancel{1}+h-\cancel{1}}}$$

$$\frac{f(1+h)-f(1)}{1+h-1} = \frac{f(1+h)-f(1)}{h}$$

Avrage rate of chang at X :

$$\frac{f(X+h)-f(X)}{h}$$



Slope of the tangent line

$=$

$f'(c)$

Ex. Find $f'(x)$ by <u>definition</u> where $f(x) = 3 - \underset{\sim}{2}x$.

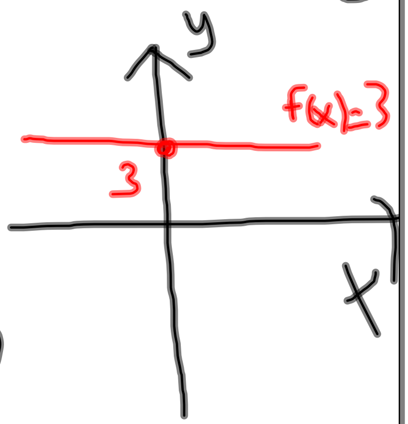
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3 - 2(x+h) - (3-2x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - \cancel{2x} - 2\cancel{h} - \cancel{3} + \cancel{2x}}{\cancel{h}} = \lim_{h \to 0} -2 = -2.$$

Differentiability $\Rightarrow$ Continuity

Continuity $\not\Rightarrow$ Differentiability

① Let $f(x) = 3$, $g(x) = -1$, $h(t) = 2/3$



$$\frac{d}{dx}\left(f(x)\right) = f'(x) = \frac{d}{dx}(3) = 0$$

$$\frac{d}{dx}\left(g(x)\right) = g'(x) = \frac{d}{dx}(1) = 0$$

$$\frac{d}{dt}\left(h(t)\right) = h'(t) = \frac{d}{dt}(2/3) = 0$$

② - Power Rule -

$$\frac{d}{dx}(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$$

$$\frac{d}{dx}(x^{5}) = 5 \cdot x^{5-1} = 5 \cdot x^4$$

$$\frac{d}{dx}(\sqrt[2]{x^1}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2}$$

$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3} \cdot x^{\frac{2}{3}-1} = \frac{2}{3} x^{-1/3}.$$

③ Constant-multiple Rule:

$$\frac{d}{dx}(3x) = 3 \cdot \frac{d}{dx}(x) = 3 \cdot 1 = 3$$

$$\frac{d}{dx}(5x^7) = 5\frac{d}{dx}(x^7) = 5 \cdot 7 \cdot x^{7-1}$$

$$= 35 \cdot x^6.$$

4 Sum and difference

$$\frac{d}{dx}\left(3x^5+x\right)=\frac{d}{dx}\left(3x^5\right)+\frac{d}{dx}(x)=3\cdot 5x^4+1$$
$$=15x^4+1.$$

$$\frac{d}{dx}\left(x^{3/4}-7\right)=\frac{d}{dx}\left(x^{3/4}\right)-\frac{d}{dx}(7)=$$
$$=\frac{3}{4}x^{\frac{3}{4}-1}-0=\frac{3}{4}\cdot x^{-1/4}.$$

Product Rule:

$$\frac{d}{dx}(f \cdot g) = (f \cdot g)' = f' \cdot g + f \cdot g'$$

① $$\frac{d}{dx}\left(x^2\cdot x^3\right) = \frac{d}{dx}\left(x^5\right) = 5\cdot x^4.$$

* Apply Product Rule:

$$\frac{d}{dx}\left(\underbrace{x^2}_{f}\cdot\underbrace{x^3}_{g}\right) = f'\cdot g + f g' = 2\cdot x^1\cdot x^3 + x^2\cdot 3x^2$$

$$= 2x^4 + 3x^4 = 5x^4$$

② Apply the product rule to find $f'(t)$, where $f(t) = t(t^2 - 4)$

$$\frac{d}{dt}\left[\underbrace{t}_{f}\underbrace{(t^2-4)}_{g}\right] = f' \cdot g + f \cdot g' =$$

$$= 1 \cdot (t^2 - 4) + t \cdot 2t$$

$$= t^2 - 4 + 2t^2 = 3t^2 - 4$$

$$\frac{d}{dt}(t^2 - 4) = \frac{d}{dt}(t^2) - \frac{d}{dt}(4) = 2t - 0 = 2t$$

$$\frac{d}{dt}(t^1) = 1 \cdot t^0 = 1 \cdot 1 = 1$$

$$\frac{d}{dx}(4) = \frac{d}{dt}(4) = \frac{d}{ds}(4) = 0.$$

$$\frac{d}{dx}(x^{1}) = 1 \cdot x^{1-1} = 1 \cdot x^{0} = 1$$

$$\frac{d}{dt}(t) = \frac{d}{dp}(p) = \frac{d}{ds}(s) = 1.$$

① Let $f(x) = \frac{1}{x} - 3x^2 + \frac{2}{\sqrt[3]{x^2}}$.

$$\frac{d}{dx}\left[\frac{1}{x} - 3x^2 + \frac{1}{\sqrt[3]{x^2}}\right] = \frac{d}{dx}\left[\frac{1}{x}\right]$$

$$- \frac{d}{dx}\left[3x^2\right] + \underbrace{\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x^2}}\right]}_{\frac{1}{x^{2/3}} = x^{-2/3}} = \frac{\cancel{0 \cdot x} - 1 \cdot 1}{x^2}$$

$$-3 \cdot 2 \cdot x + \left(-\frac{2}{3} \cdot x^{-\frac{2}{3}-1}\right)$$

$$= -\frac{1}{x^2} - 6x - \frac{2}{3} \cdot x^{-5/3}.$$

$$\boxed{\frac{d}{dx}\left(x^{-1}\right) = -1 \cdot x^{-2}}$$

② Let $f(x) = \underbrace{(\sqrt{x}+1)}_{h(x)}\underbrace{(2x - x^2)}_{g(x)}$

(Apply the product rule)

$f' = h'(x)\cdot g(x) + h(x)\cdot g'(x)$

$= \frac{d}{dx}(x^{1/2}+1)\cdot(2x-x^2) + (\sqrt{x}+1)\frac{d}{dx}(2x-x^2)$

$= \frac{1}{2}\cdot x^{\frac{1}{2}-1}\cdot(2x-x^2) + (\sqrt{x}+1)\cdot(2-2x)$

$= \frac{1}{2}x^{-1/2}(2x-x^2) + (\sqrt{x}+1)(2-2x)$

$= x^{1/2} - \frac{1}{2}x^{3/2} + 2x^{1/2} - 2x^{3/2}$

$+ 2 - 2x$

$= 3x^{1/2} - \frac{5}{2}x^{3/2} + 2 - 2x$

(3) Let $f(x) = \dfrac{x - 3x^2}{x+1}$.

Apply the quotient rule to find f'.

$$f' = \frac{\frac{d}{dx}(x-3x^2)\cdot(x+1) - \frac{d}{dx}(x+1)\cdot(x-3x^2)}{(x+1)^2}$$

$$= \frac{(1-6x)(x+1) - 1\cdot(x-3x^2)}{(x+1)^2}$$

$$= \frac{x+1-6x^2-6x-x+3x^2}{(x+1)^2}$$

$$= \frac{-3x^2-6x+1}{(x+1)^2} = \frac{-3x^2-6x+1}{x^2+2x+1}.$$

$$R(p) = p(12 - p - p^2)$$
$$= \underbrace{12p} - p^2 - p^3.$$
$$R'(p) = 12 - 2p - 3p^2.$$
$$R''(p) = 0 - 2 - 6p$$
$$= -2 - 6p$$
$$R'''(p) = 0 - 6 = -6$$
$$R^{(iv)}(p) = 0.$$

Mar 8-10:08

$$\varphi(t) = -t^3 + 6t^2 + 24t$$

$$\varphi'(t) = -3t^2 + 12t + 24$$

$$\varphi''(t) = -6t + 12 + 0$$

$$\varphi'(3) = -3 \cdot 3^2 + 12 \cdot 3 + 24 \qquad \varphi''(3) = -6 \cdot 3 + 12$$

Find the fifth order derivative

of $y = 1/x$.

$f(x) = \frac{1}{x} = x^{-1}$.

$f'(x) = -1x^{-2} = -x^{-2}$

$f''(x) = 2x^{-3}$.

$f'''(x) = -6x^{-4}$.

$f^{(iv)}(x) = 24x^{-5}$

$f^{(v)}(x) = -120x^{-6}$

$\vdots \qquad \vdots \qquad \vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

Let $y = f(x) = \sqrt{x} - \frac{1}{2x} + \frac{x}{\sqrt{2}}$.

$y'''\left(\frac{d^3y}{dx^3}\right) = ?$

$$y = x^{1/2} - \frac{1}{2}\cdot x^{-1} + \frac{1}{\sqrt{2}}\cdot x \qquad \frac{1}{2}\cdot -2$$

$$y' = \frac{1}{2}\cdot x^{-1/2} + \frac{1}{2}\cdot x^{-2} + \frac{1}{\sqrt{2}} \qquad \frac{1}{2}\cdot\frac{-1}{2}$$

$$y'' = -\frac{1}{4}\cdot x^{-3/2} - 1x^{-3} + 0 \qquad = -\frac{1}{4}$$

$$y''' = \frac{3}{8}x^{-5/2} + 3x^{-4}$$

Chain Rule:

Let $f = f(u)$ and $u = u(x)$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$= \left(\text{Derivative of outside}\right) \times$

$\left(\text{Derivative of inside}\right)$

Mar 11-10:10

Recall $(a-b)^2 = a^2-2ab+b^2$

Example (Chain Rule)

Let $y=f(x)=(2x-1)^2$. $y' = \frac{dy}{dx} = ?$

First way: $y=(\underbrace{2x-1}_{u})^2 \Rightarrow y=4x^2-4x+1$

$\Rightarrow y' = \underline{\underline{8x-4}}$.

Second: Let $y=(2x-1)^2$, $u = 2x-1$, $y = u^2$

By chain Rule: $y' = 2u \cdot 2 = 4u = 4(2x-1)$

$= \underline{\underline{8x-4}}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Chain Rule.

Example: Let $y = x^3 - x$ and $x = 1 - 3t$

a) $\frac{dy}{dt} = ?$ b) $y'(0) = \frac{dy}{dt}(0)$.

---

a) • $y = (1-3t)^3 - (1-3t)$ (exercise)

• $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 - 1) \cdot (-3) = \left(3 \cdot (1-3t)^2 - 1\right)$

$x \quad \cdot -3$

b) $y'(0) = \left(3 \cdot (1 - 3\cdot 0)^2 - 1\right) \cdot -3$

$= -6$

Let $y = \dfrac{3}{(x^2+1)^2}$. $y'(x) = \dfrac{dy}{dx} = ?$.

First way: $y = \dfrac{3}{x^4+2x^2+1}$.

Apply the quotient rule (exercise)

Second way (Chain rule)

$y = 3\cdot(x^2+1)^{-2}$

$y' = 3\cdot -2\cdot(x^2+1)^{-3}\cdot 2x$

$= -12x(x^2+1)^{-3}$.

Mar 11-10:29

$$P(w) = \sqrt{3w^2 + 30w}$$

$$P(w) = \left(3w^2 + 30w\right)^{1/2}$$

$$= \frac{1}{2}\left(3w^2 + 30w\right)^{-1/2} \cdot \left(6w + 30\right)$$

$$= \frac{6w + 30}{2\left(3w^2 + 30w\right)^{1/2}}$$

Mar 11-10:37

$\underline{\text{example}}$: let $y = \left(\sqrt{x} + x\right)^{1/3}$

$\Rightarrow \quad y = \left(x^{1/2} + x\right)^{1/3}.$

$y' = \frac{1}{3} \cdot \left(x^{1/2} + x\right)^{-2/3} \cdot \left(\frac{1}{2}x^{-1/2} + 1\right)$

$= ($

Mar 11-10:43

Let $2x^2+y^2=8$.

a) $y^2 = 8-2x^2$

$\Rightarrow y = \pm\sqrt{8-2x^2} = \pm(8-2x^2)^{1/2}$

$\Rightarrow \boxed{y' = \pm\frac{1}{2}(8-2x^2)^{-1/2}\cdot -4x}$

b) Let $2x^2+y^2=1$.

$\frac{d}{dx}(y^2_{(x)}) = 2y\cdot y'$.

$\frac{d}{dy}(y^2) = 2y$

$4x + 2yy' = 0$

$2yy' = -4x \Rightarrow y' = \frac{-4x}{2y}$

$\Rightarrow \boxed{y' = -2\frac{x}{y}}$

Mar 13-10:22

Example: Find $\frac{dy}{dx}$ by implicit differentiation and by differentiating an explicit formula for y. In each case, show that the answers are the same given

a) * $\boxed{xy=4} \Rightarrow y = \frac{4}{x} = 4 \cdot x^{-1}$

$$\boxed{y' = -4x^{-2}}$$

* Differentiate both sides to $xy=4$

$1 \cdot y + xy' = 0$

$\Rightarrow xy' = -y \Rightarrow y' = -\frac{y}{x} = -\frac{4x^{-1}}{x}$

$$\boxed{= -4x^{-2}}$$

b) Let $xy + 2y = x^2$.

* Find an explicit formula for y:

$y(x+2) = x^2 \Rightarrow y = \frac{x^2}{x+2}$

By quotient rule,

$$y' = \frac{2x(x+2) - 1\cdot(x^2)}{(x+2)^2}$$

$$\boxed{y' = \frac{x^2+4x}{(x+2)^2}}$$

* $xy + 2y = x^2$. By implicit differentiation;

$y + xy' + 2y' = 2x$

$\Rightarrow y'(x+2) = 2x - y$

$$\Rightarrow y' = \frac{2x-y}{x+2} = \frac{2x - \frac{x^2}{x+2}}{x+2}$$

$$\boxed{= \frac{x^2+4x}{(x+2)^2}}$$

$$\frac{d}{dx}\left(x \cdot \sqrt{2x+1}\right)$$

$$= \frac{d}{dx}(x) \cdot \sqrt{2x+1} + x \cdot \frac{d}{dx}\Big(\underbrace{\sqrt{2x+1}}_{=(2x+1)^{1/2}}\Big)$$

$$= 1 \cdot \sqrt{2x+1} + x \cdot \frac{1}{2} \cdot (2x+1)^{\frac{1}{2}-1} \cdot 2$$

$$= \sqrt{2x+1} + x(2x+1)^{-1/2}.$$

$$\frac{d}{dx}\left(y^3_{(x)}\right) = \underbrace{3 \cdot y^2}_{\text{Deivative of outside}} \cdot \underbrace{y'}_{\text{Derivative of inside}}$$

$$\frac{d}{dx}\left(x^2 y^2_{(x)}\right) = \frac{d}{dx}(x^2) \cdot y^2 + x^2 \frac{d}{dx}(y^2)$$

$$= 2x \cdot y^2 + x^2 \cdot \underbrace{2y}_{\text{Derivative of outside}} \cdot \underbrace{y'}_{\text{Derivative of inside}}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}[y]^2 = 2y \cdot y'$$

Let $y = 2x+1$

$$\frac{d}{dx}\left((2x+1)\right)^2 = 2(2x+1)^{2-1} \cdot 2$$

Relative Extrema

A relative maximum of a function *f* is located at a value M such that $f(x) \leq f(M)$ for all values of *x* on an interval $a < M < b$.

A relative minimum of a function *f* is located at a value m such that $f(x) \geq f(m)$ for all values of *x* on an interval $a < m < b$.

Critical Points

Let *c* be a value in the domain of *f*.

If $f'(c) = 0$ or $f'(c)$ is undefined, then we say *f* has a critical point at *c*.

Relative extrema can only occur at critical points.

Quick Example

The profit function in Example 1 has critical points where

$$P'(x) = 9 - \tfrac{3}{2}x^{1/2} = 0,$$

and where P'(x) is undefined. So P has critical points at

$$x = 36$$

Note from the graph that the function has a relative relative maximum at x=36.

EXAMPLE 2 Finding and Classifying Critical Numbers

Find all critical numbers and classify them as a relative maximum, relative minimum, or neither for

$$f(x) = 4x + \frac{2}{x^2}.$$

EXAMPLE 2 Finding and Classifying Critical Numbers

SOLUTION

Relative extrema can only take place at critical points (but not necessarily all critical points end up being extrema!).

Thus we need to find critical points of *f*. In other words, values of *x* so that $f'(x)=0$ or $f'(x)$ is undefined.

EXAMPLE 2 Finding and Classifying Critical Numbers

SOLUTION

$$0 = f'(x) = \frac{d}{dx}\left[4x + \frac{2}{x^2}\right]$$

$$0 = 4 + 2 \cdot -2x^{-3}$$

$$-4 = -4x^{-3}$$

$$1 = x^{-3}$$

$$x = 1.$$

Now we need to consider critical points due to the derivative being undefined.

EXAMPLE 2 **Finding and Classifying Critical Numbers**

SOLUTION

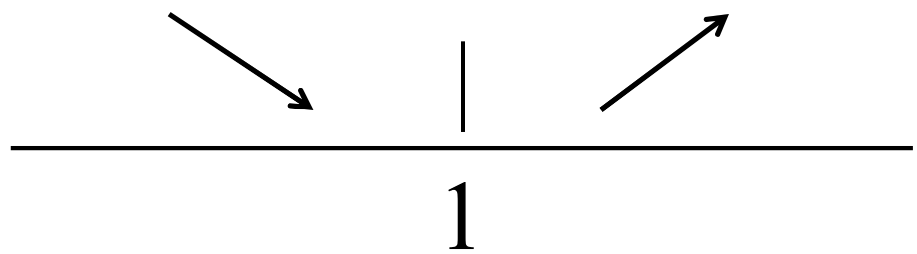
$f'(x) = 4 - 4x^{-3}$ is undefined when $x = 0$. However, it is very important to note that 0 cannot be the location of a critical point, because *f* is undefined at 0.

In other words, no critical point of a function can exist if no *point* on *f* exists.

EXAMPLE 2 Finding and Classifying Critical Numbers

SOLUTION

Now we classify the critical number we've found at $x = 1$:



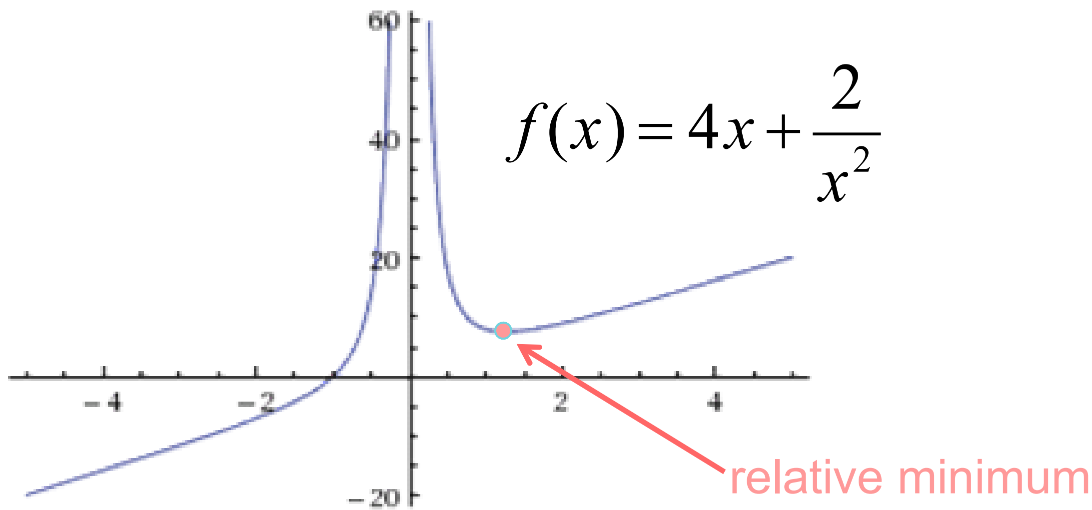
Calculating the derivative at a test point to the left of 1 (e.g. x=0.5), we find $f'(0.5) = 4 - 4(0.5)^{-3} = -28,$ so *f* is decreasing.

Similarly, $f'(2) = 4 - 4(2)^{-3} = 3.5,$ so *f* is increasing.

EXAMPLE 2 Finding and Classifying Critical Numbers

SOLUTION

From our diagram it appears that *f* has a relative minimum at $x = 1$. A graph of the function corroborates this claim.



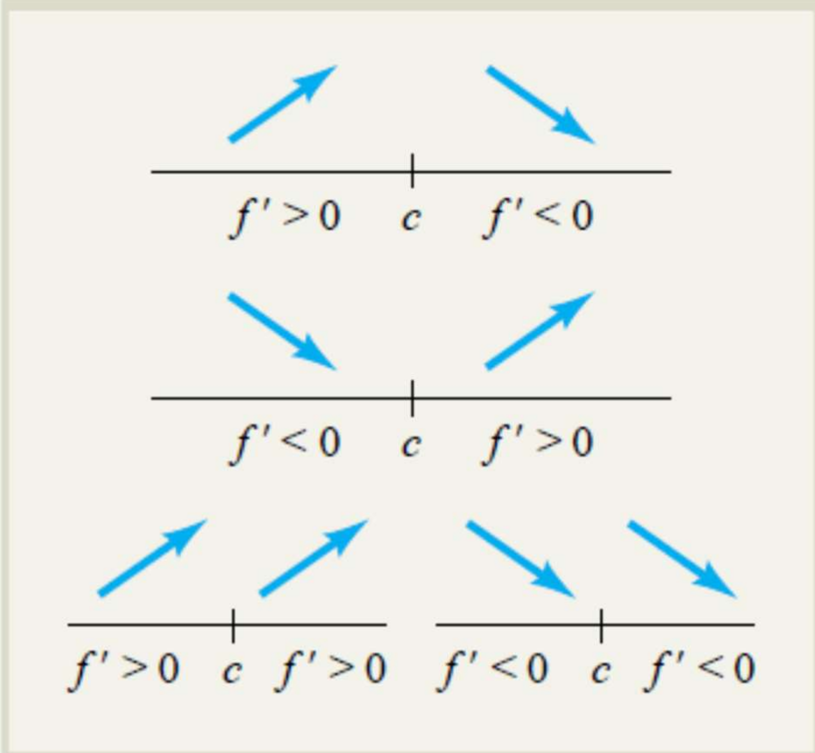
The first Derivative Test

The First Derivative Test for Relative Extrema ■ Let c be a critical number for $f(x)$ [that is, $f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ does not exist]. Then the critical point $P(c, f(c))$ is

a **relative maximum** if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c

a **relative minimum** if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c

not a relative extremum if $f'(x)$ has the same sign on both sides of c



EXAMPLE 3.1.3

Find all critical numbers of the function

$$f(x) = 2x^4 - 4x^2 + 3$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Solution

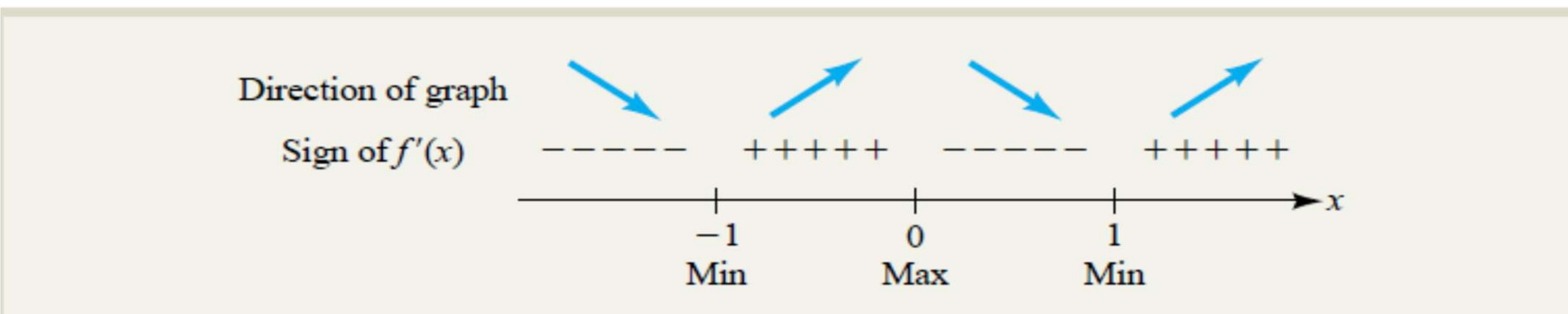
The polynomial $f(x)$ is defined for all x, and its derivative is

$$f'(x) = 8x^3 - 8x = 8x(x^2 - 1) = 8x(x - 1)(x + 1)$$

Since the derivative exists for all x, the only critical numbers are where $f'(x) = 0$; that is, $x = 0$, $x = 1$, and $x = -1$. These numbers divide the x axis into four intervals, on each of which the sign of the derivative does not change; namely, $x < -1$, $-1 < x < 0$, $0 < x < 1$, and $x > 1$. Choose a test number c in each of these intervals (say, -5, $-\frac{1}{2}$, $\frac{1}{4}$, and 2, respectively) and evaluate $f'(c)$ in each case:

$$f'(-5) = -960 < 0 \qquad f'\left(-\frac{1}{2}\right) = 3 > 0 \qquad f'\left(\frac{1}{4}\right) = -\frac{15}{8} < 0 \qquad f'(2) = 48 > 0$$

Thus, the graph of f falls for $x < -1$ and for $0 < x < 1$, and rises for $-1 < x < 0$ and for $x > 1$, so there must be a relative maximum at $x = 0$ and relative minima at $x = -1$ and $x = 1$, as indicated in this arrow diagram.



A Procedure for Sketching the Graph of a Continuous Function $f(x)$ Using the Derivative $f'(x)$

Step 1. Determine the domain of $f(x)$. Set up a number line restricted to include only those numbers in the domain of $f(x)$.

Step 2. Find $f'(x)$ and mark each critical number on the restricted number line obtained in step 1. Then analyze the sign of the derivative to determine intervals of increase and decrease for $f(x)$ on the restricted number line.

Step 3. For each critical number c, find $f(c)$ and plot the critical point $P(c, f(c))$ on a coordinate plane, with a "cap" ⌢ at P if it is a relative maximum (↗ ↘), or a "cup" ⌣ if P is a relative minimum (↘ ↗). Plot intercepts and other key points that can be easily found.

Step 4. Sketch the graph of f as a smooth curve joining the critical points in such a way that it rises where $f'(x) > 0$, falls where $f'(x) < 0$, and has a horizontal tangent where $f'(x) = 0$.

EXAMPLE 3.1.4

Sketch the graph of the function $f(x) = x^4 + 8x^3 + 18x^2 - 8$.

Solution

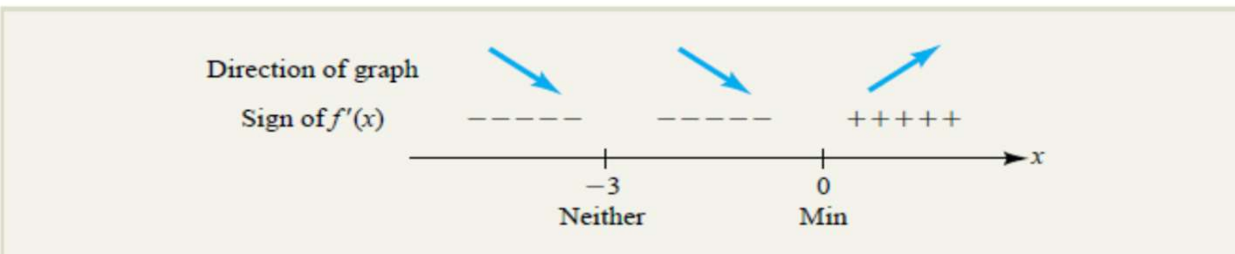
Since $f(x)$ is a polynomial, it is defined for all x. Its derivative is

$$f'(x) = 4x^3 + 24x^2 + 36x = 4x(x^2 + 6x + 9) = 4x(x + 3)^2$$

Since the derivative exists for all x, the only critical numbers are where $f'(x) = 0$; namely, at $x = 0$ and $x = -3$. These numbers divide the x axis into three intervals, on each of which the sign of the derivative $f'(x)$ does not change; namely, $x < -3$, $-3 < x < 0$, and $x > 0$. Choose a test number c in each interval (say, -5, -1, and 1, respectively), and determine the sign of $f'(c)$:

$$f'(-5) = -80 < 0 \qquad f'(-1) = -16 < 0 \qquad f'(1) = 64 > 0$$

Thus, the graph of f has horizontal tangents where x is -3 and 0, and the graph is falling (f decreasing) in the intervals $x < -3$ and $-3 < x < 0$ and is rising (f increasing) for $x > 0$, as indicated in this arrow diagram:



Interpreting the diagram, we see that the graph falls to a horizontal tangent at $x = -3$, then continues falling to the relative minimum at $x = 0$, after which it rises indefinitely. We find that $f(-3) = 19$ and $f(0) = -8$. To begin your sketch, plot a "cup" at the critical point $(0, -8)$ to indicate that a relative minimum occurs there (if it had been a relative maximum, you would have used a "cap"), and a "twist" at $(-3, 19)$ to indicate a falling graph with a horizontal tangent at this point. This is shown in the preliminary graph in Figure 3.9a. Finally, complete the sketch by passing a smooth curve through the critical points in the directions indicated by the arrows, as shown in Figure 3.9b.

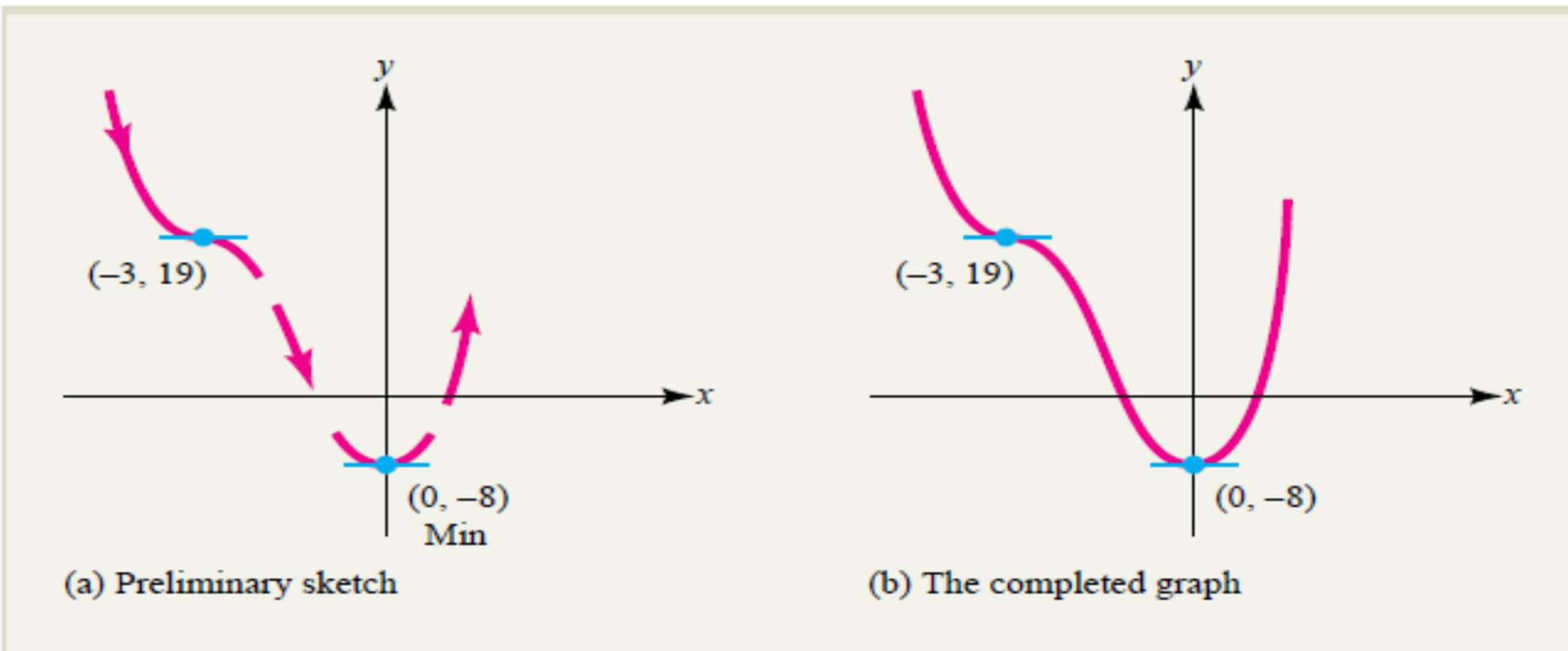


FIGURE 3.9 The graph of $f(x) = x^4 + 8x^3 + 18x^2 - 8$.

Example: Let $f(x) = 2x^4 - 4x^2 + 3$.
Find all critical points and classify them.

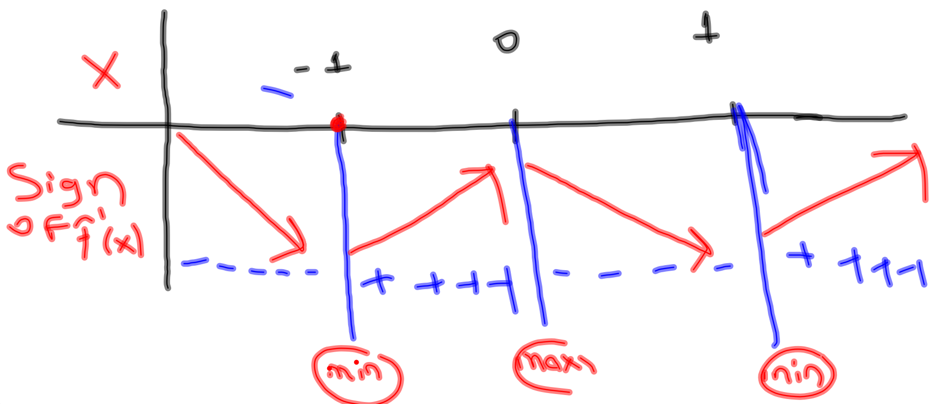
Solution:

$$f'(x) = 8x^3 - 8x = 0$$

$$8x(x^2 - 1) = 0$$

$$8x(x-1)(x+1) = 0$$

$$\Rightarrow x = 0,\ x = 1,\ x = -1 \text{ (critical points)}$$



Relative maxima occurs at $x = 0$.

Relative minima occurs at $x = -1$ and $x = 1$.

Relative maxima: $f(x) = 2x^4 - 4x^2 + 3$; Put $x = 0$

$= 3$ $\qquad \boxed{(0,3)}$

Relative minima: Put $x = -1$ and $x = 1$

$y = 1 \qquad y = 1$

$\boxed{(-1,1)} \qquad \boxed{(1,1)}$

Example. Sketch the graph of the function $f(x) = x^4 + 8x^3 + 18x^2 - 8$.

Solution: S1) The domain $= \mathbb{R} \cdot (-\infty, \infty)$

S2) $f'(x) = 4x^3 + 24x^2 + 36x = 0$

$4x(x^2 + 6x + 9) = 4x(x+3)^2 = 0$

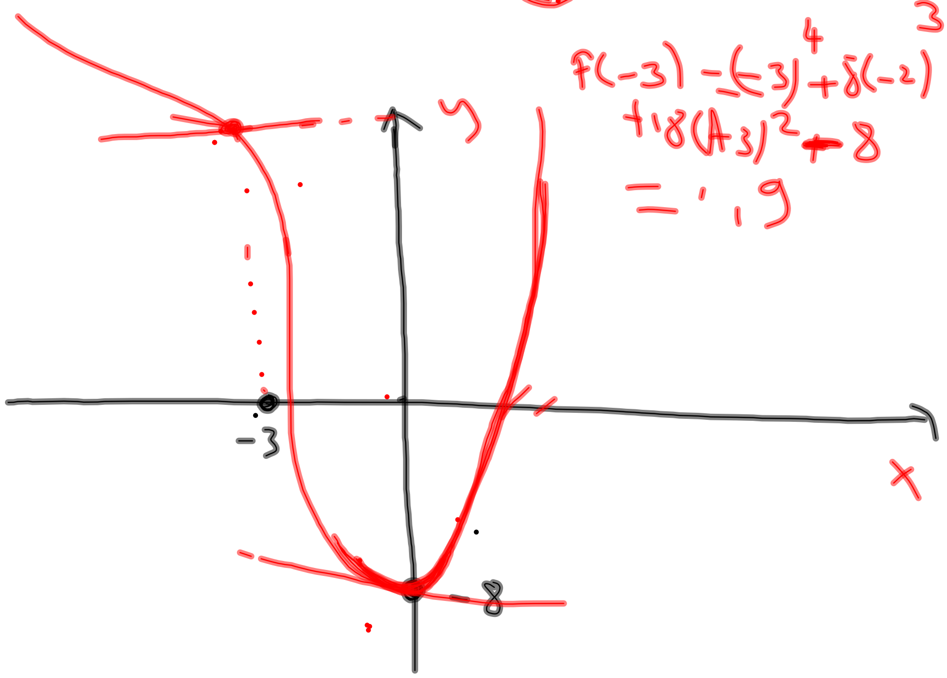
$\boxed{x = 0 \ , \ x = -3}$

		-3		0	
Sign of $f'(x)$	$-$		$-$		$+$

-3: neither maxima nor minima

0: Relative minima $f(0) = -8$ $(0, -8)$

$f(-3) = (-3)^4 + 8(-3)^3 + 18(-3)^2 - 8 = 19$

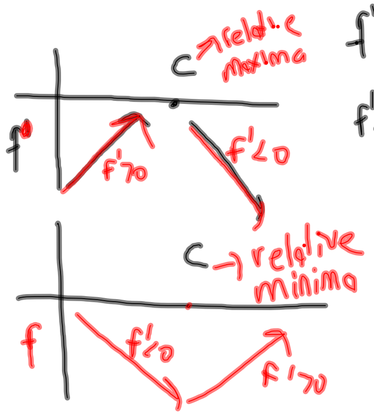


So far, in ch. 3, you learned that

- Critical points: $f(x)$ is defined, but either $f'(x)=0$ or $f'(x)$ is undefined

$f'>0 \Rightarrow f$ is increasing

$f'<0 \Rightarrow f$ is decreasing



Second derivative test for concavity:

S1) Find the points c such that $f(c)$ is defined but $f''(c)=0$ or $f''(c)$ does not exist

S2) if $f''(x)>0$, $a<x<b$ f is concave up. $\cup$

if $f''(x)<0$, $a<x<b$ f is concave down $\cap$

Ex Let $f(x) = 2x^6 - 5x^4 + 7x - 3$

$$f' = 12x^5 - 20x^3 + 7$$

$$f'' = 60x^4 - 60x^2 = 0 \quad \boxed{60x^2(x^2-1)} = 0$$

$$= 60x^2(x-1)(x+1) = 0 \qquad x = 0$$

$$\boxed{x=0 \quad x=1 \quad x=-1}$$

	$-\infty$	-1	0	1	∞
Sign of f''	+	−	−	+	
	$\cup$	$\cap$	$\cap$	$\cup$	

Example: (Review example)

Let $f(x) = 3x^4 - 2x^3 - 12x^2 + 18x + 15$

a) Find intervals where f is decreasing and increasing

b) Find all relative extrema.

c) Find intervals where f is concave up and concave down

d) Find all inflection points

e) Sketch the graph.

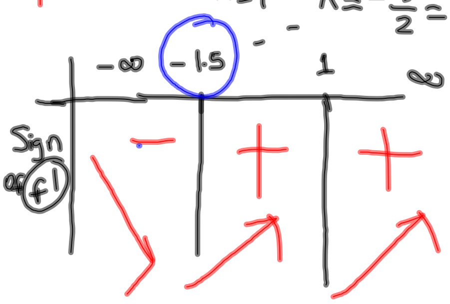
---

a) $f' = 12x^3 - 6x^2 - 24x + 18 = 0$

$\underline{\text{(exercise)}} = 6(x-1)^2(2x+3) = 0$

$x = 1 \quad x = -\frac{3}{2} = -1.5$

a)



b) at $x = -1.5$ there is a relative minimum $f(-1.5) = -17$.

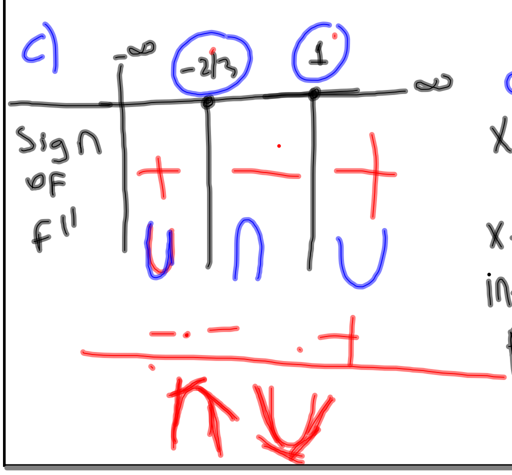
c) $f''(x) = 36x^2 - 12x - 24 = 0$

$= 12(3x^2 - x - 2) = 0$

$= 12(x-1)(3x+2) = 0$

$\boxed{x = 1} \quad \boxed{x = -2/3}$

c)



d) $f(-\frac{2}{3}) = -11$

$x = -2/3$

$x = 1 \quad f(1) = 22$

are inflection points

Example (Relative extrema)

Let $f(x) = 2x^3 + 3x^2 - 12x - 7$.

Find all critical points and classify them as a relative min., or max.

By first derivative: $6x^2 + 6x - 12 = 0$

$$6(x^2 + x - 2) = 6(x+2)(x-1)$$

$$\boxed{x = -2 \quad x = 1}$$

x	$-\infty$	-2 max.		1 min.	∞
Sign of f'	$+$		$-$		$+$
f	$\nearrow$		$\searrow$		$\nearrow$

Second derivative test:

$$f''(x) = 12x + 6$$

$f''(1) = 12 \cdot 1 + 6 = 18 > 0$, Relative min. occurs at $x = 1$

$f''(-2) = 12 \cdot -2 + 6 = -18 < 0$. Relative max. occurs at $x = -2$

Example (Sketch the graph)

Let $f(x)=\frac{1}{2}x^4-\frac{2}{3}x^3-2x^2+3$.

① Domain $=\mathbb{R}$

② y intercept: $x=0,\ y=3$ $\quad(0,3)$

x intercept: $y=0 \Rightarrow \frac{1}{2}x^4-\frac{2}{3}x^3-2x^2+3=0$

$x=1.2 \quad x=2.5$ (by calculators)

$(1.2,0)$ and $(2.5,0)$ are x-intercepts.

③ no asymptotes

④ $f' = 2x^3-2x^2-4x = 2x(x^2-x-2)=0$

$= 2x(x-2)(x+1)=0$

$\Rightarrow \boxed{x=0,\ x=-1,\ x=2}$

x	$-\infty$	-1 min.		0 max.		2 min.	∞
Sign of f'	$-$		$+$		$-$		$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

At $x=-1$, $x=2$ relative min. occurs

$\Downarrow$

$y=?\quad y=?$

At $x=0$, relative max. occurs

$\Downarrow$

$y=$

⑤

$y''=6x^2-4x-4=0 \qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$3x \quad 3x \qquad -2,\ 2 \qquad 4,\ -1 \qquad 6 \qquad -4$

$x=-0.5 \qquad x=1.2$

x	$-\infty$	-0.5 inflection		1.2 inflection	∞
sign of f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

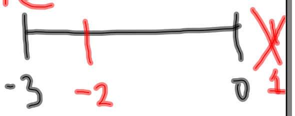
Example: Let $f(x)=2x^3+3x^2-12x-7$, Find absolute max. and min. on $[-3,0]$

Solution: $f'(x)=6x^2+6x-12=0$

$$=6(x^2+x-2)=0$$

$$=6(x+2)(x-1)=0$$

$x=-2$, $\cancel{x=1}$ ignore



$$f(-2)=2(-2)^3+3(-2)^2-12(-2)-7$$

$$=-16+12+24-7=13$$

$\boxed{f(-2)=13}$

$$f(0)=2\cdot 0^3+3\cdot 0^2-12\cdot 0-7=-7$$

$\boxed{f(0)=-7}$ $\boxed{f(-3)=2}$

$$f(-3)=2(-3)^3+3(-3)^2-12(-3)-7$$

$$=-54+27+36-7=2$$

$\boxed{f(-3)=2}$

The absolute max. of the function is 13 and it occurs at $x=-2$.

The absolute min. of the function is -7 and it occurs at $x=0$.

Example. Find absolute extrema of $f(x) = \dfrac{x^2}{x-2}$ on $[-3, 1]$.

Solution: $f'(x) = \dfrac{\overbrace{2x\cdot(x-2) - 1\cdot x^2}^{2x^2-4x-x^2}}{(x-2)^2} = \dfrac{x^2-4x}{(x-2)^2}$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$x^2 - 4x = x(x-4) = 0$

$\boxed{x=0}$ $\quad x=4$ ignore $\quad x=2$ ignore

$f(0) = \dfrac{0}{0-2} = 0 \Rightarrow \boxed{f(0)=0}$ largest

$f(-3) = \dfrac{(-3)^2}{-3-2} = -\dfrac{9}{5} \Rightarrow \boxed{f(-3) = -\dfrac{9}{5}}$ smallest

$f(1) = \dfrac{1^2}{1-2} = -1 \Rightarrow \boxed{f(1) = -1}$

Conclusion: The absolute max. is 0, it occurs at $x=0$.
Absolute min. is $-9/5$, it occurs at $x=-3$.

Maximum Profit

$$P(x) = R(x) - C(x)$$

$$P'(x) = R'(x) - C'(x) = 0$$

$$\boxed{R'(x) = C'(x)}$$

marginal Revenue = Marginal Cost

$$P''(x) = R''(x) - C''(x) < 0$$

$$\boxed{R''(x) < C''(x)}$$

$$2x + 1 = 50 - 3x^2$$

$$\Rightarrow \underset{a}{-3}x^2 \overset{b}{-2}x + \overset{c}{49} = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4\cdot(-3)\cdot 49}}{-6}$$

$$\cancel{\boxed{x = -4.3}}, \quad \boxed{x = 3.72}$$

Exponential Functions:

$$f(x) = b^{x}$$

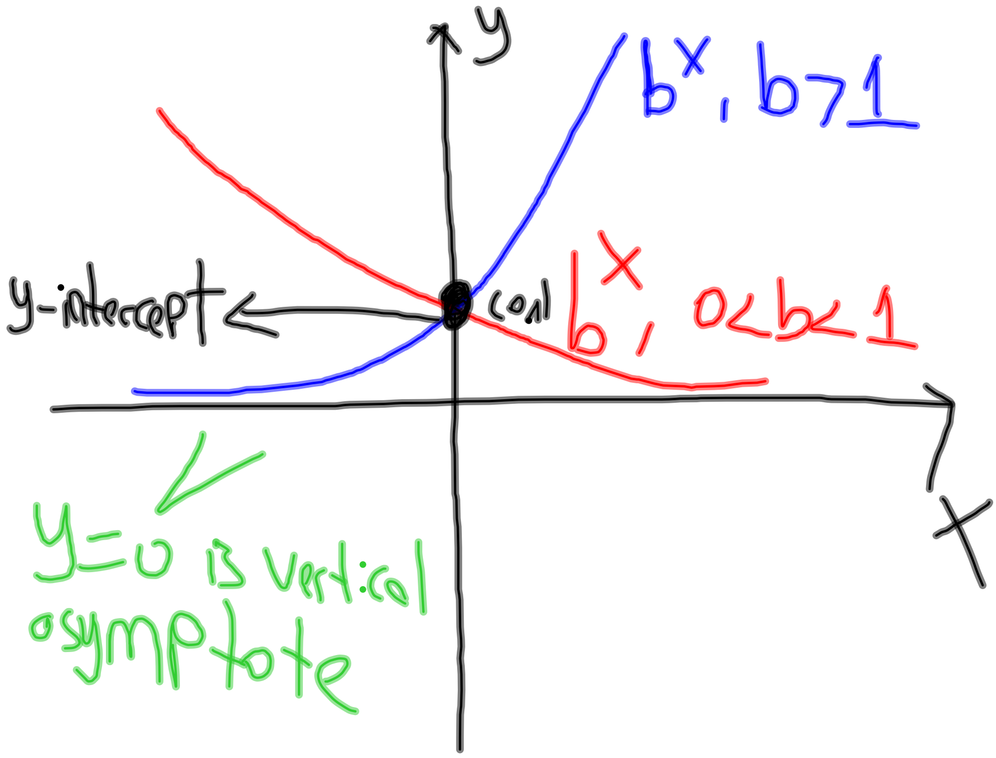
x = power

b = Base

$x \in (-\infty, \infty) = \mathbb{R}$ = Domain

$b \neq 1$, $b > 0$

Range: $(0, \infty)$



no x-intercept why?

$b^x = 0$

Exponential Function:

$$y = b^x, \quad b > 0, \quad b \neq 1, \quad x \in \mathbb{R}$$

Logarithmic Function

$$y = \log_b x \quad b > 0, b \neq 1, x > 0$$

$$y = \log_b x \iff b^y = x$$

$$\log_4 64 = x = 3.$$

$$4^x = 64 \Rightarrow x = 3$$

$$\log_5 \sqrt{125} = x$$

$$5^x = \sqrt{125} = \sqrt[2]{5^3} = 5^{3/2}$$

Bases are the same

$$x = 3/2$$

Section objective: to study logarithmic properties:

$$y = \log_b x, \quad b > 0, \; b \neq 1, \; x > 0$$

Properties: Let a, b, c are positive real numbers, $b \neq 1$, $n \in \mathbb{R}$

1. $\log_b (a \cdot c) = \log_b a + \log_b c$. Product rule
2. $\log_b \left(\frac{a}{c}\right) = \log_b a - \log_b c$. Quotient rule
3. $\log_b a^n = n \cdot \log_b a$. Power Rule
4. $\log_b 1 = 0$, $\log_b b = 1$.
5. $\log_b c = \log_b a \iff c = a$ (one-to-one property)
6. $\log_b b^x = x$, $b^{\log_b x} = x$ ($\log_b x$ and b^x are mutually inverse)

$\log x := \log_{10} x$.

$\ln x := \log_e x$.

$e \approx 2.72$ is an irrational number

Change of base:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Examples :

① Rewrite each expression as a single logarithm :

a) $2\log_b x + \frac{1}{2}\log_b(x+4)$

$= \log_b x^2 + \log_b (x+4)^{1/2}$, Power rule

$= \log_b \left(x^2 \cdot (x+4)^{1/2}\right)$, product rule

b) $4\log_b(x+2) - 3\log_b(x-5)$

$= \log_b (x+2)^4 - \log_b (x-5)^3$, power rule

$= \log_b \frac{(x+2)^4}{(x-5)^3}$. Quotient rule

② Express the following logarithms in terms of logarithms of x, y, z :

a) $\log_b(xy^2) = \log_b x + \log_b y^2$

(Product rule)

(Power Rule) $= \log_b x + 2\log_b y$

b) $\log_b \frac{x^2\sqrt{y}}{z^5}$.

$= \log_b(x^2\sqrt{y}) - \log_b(z^5)$ quotient rule

$= (\log_b x^2 + \log_b \sqrt{y}) - \log_b(z^5)$ Product rule

$= 2\log_b x + \frac{1}{2}\log_b y - 5\log_b z$

Caution : • Dont confuse $\frac{\log 9}{\log 2}$ and $\log \frac{9}{2}$.

$\log \frac{9}{2} = \log 9 - \log 2$.

• Do not confuse $\log 9 \cdot \log 2$, $\log(9.2)$

$\log(9.2) = \log 9 + \log 2$.

Change of Base Formula:

$$\log_a b = \frac{\log_c b}{\log_c a} \underset{\boxed{c=e}}{=} \frac{\ln b}{\ln a}$$

For Example

$\log_e x = \ln x$

① $\log_4 7 = \dfrac{\ln 7}{\ln 4}$

② $\log_2 5 = \dfrac{\ln 5}{\ln 2}$

$= \dfrac{\log 5}{1}$

$\log_{10} x$

$= \log x$

[Example 2 in Section 4.2 is left as an exercise]

Apr 15-10:06

Objective: Derivatives of exponential and logarithmic functions

$$\frac{d}{dx}(e^x) = e^x.$$

Example. $\frac{d}{dx}(\underbrace{e^x}_{\text{First}}\underbrace{\sqrt{1-x}}_{\text{Second}}) = ?$

Product rule

$$\frac{d}{dx}(e^x)\cdot(1-x)^{1/2} + e^x\cdot\frac{d}{dx}(1-x)^{1/2}$$

$$= e^x\cdot(1-x)^{1/2} + e^x\cdot\frac{1}{2}(1-x)^{-1/2}\cdot -1$$

$$= e^x\left[(1-x)^{1/2} - \frac{1}{2}(1-x)^{-1/2}\right]$$

$$\frac{d}{dx}(e^u) = e^u\cdot u'(x)$$

(Chain rule for e^u)

Example. Let $f(x) = e^{x^2-x}$.

a) Find $\frac{d}{dx}(f(x)) = ?$

b) Find the equation of the tangent line to $f(x)$ at $x=0$.

Solution: a) $\frac{d}{dx}\left(e^{x^2-x}\right) = e^{x^2-x}\cdot(2x-1)$.

b) Point-slope:

$$y - y_0 = m(x - x_0), \quad y_0 = f(x_0)$$

$y_0 = e^{0^2-0} = e^0 = 1$

y_0: ? → 1

m: ? → derivative at $x_0=0$ $= e^0\cdot -1 = -1$

x_0: ? → 0

$y - 1 = -1\cdot x$

$$\boxed{y = -x+1}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example. $\frac{d}{dx}\left(\frac{x^2}{1+\ln x}\right) = ?$

$$= \frac{2x(1+\ln x) - \left(\frac{1}{x}\cdot x^2\right)}{(1+\ln x)^2}$$

$$= \frac{2x + 2x\ln x - x}{(1+\ln x)^2} = \frac{x+2x\ln x}{(1+\ln x)^2}.$$

$$\frac{d}{dx}(\ln u) = \frac{u'}{u} \quad \text{(Chain Rule)}$$

For $u = x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Example: $\frac{d}{dx}\left(\ln(x^2+1)\cdot 3x\right) = ?$

Product Rule

$$\frac{2x}{x^2+1}\cdot 3x + \ln(x^2+1)\cdot 3$$

$$= \frac{6x^2}{x^2+1} + 3\ln(x^2+1)$$

$$\boxed{\begin{aligned} \frac{d}{dx}(b^x) &= \ln b\cdot b^x \\ \frac{d}{dx}(\log_b x) &= \frac{1}{x\cdot\ln b} \end{aligned}}$$

$b > 0$, $b \neq 1$.

Objective: Logarithmic differentation

Example. $y=f(x)=\underbrace{3x^3 e^{x-2}}_{\text{First Term}=u}\;\underbrace{\sqrt[3]{x^2+1}}_{\text{Second term}=v}$.

Calculate $y' = ?$

Solution:

method 1 (Product rule)

$$y' = \underbrace{\frac{d}{dx}\left(3x\,e^{x-2}\right)}_{\text{Product rule}} \cdot (x^2+1)^{1/3} + 3x\,e^{x-2} \cdot \underbrace{\frac{d}{dx}\left(x^2+1\right)^{1/3}}_{\text{Chain Rule}}$$

(left as an exercise) - continue

method 2: (logarithmic diff.)

$$y = 3x^3 e^{x-2}(x^2+1)^{1/3}$$

$\boxed{\ln(a.b.c) = \ln a + \ln b + \ln c}$

S1) Take $\ln$ both sides:

$$\ln y = \ln\left(3x^3 e^{x-2}(x^2+1)^{1/3}\right)$$

S2)

$$\ln y = \ln 3 + \ln x^3 + \ln e^{x-2} + \ln(x^2+1)^{1/3}$$

$$\ln y = \ln 3 + 3\ln x + (x-2)\underbrace{\ln e}_{=1} + \frac{1}{3}\ln(x^2+1)$$

$$\boxed{\ln y = \ln 3 + 3\ln x + x - 2 + \frac{1}{3}\ln(x^2+1)}$$

S3) Take derivative both sides:

$$(\ln u)' = \frac{u'}{u}.$$

$$\frac{y'}{y} = \frac{3}{x} + 1 + \frac{1}{3}\cdot\frac{2x}{x^2+1}$$

$$y' = y\left(\frac{3}{x} + 1 + \frac{2x}{3x^2+3}\right)$$

$$\boxed{y' = 3x^3 e^{x-2}(x^2+1)^{1/3} \cdot \left(\frac{3}{x} + 1 + \frac{2x}{3x^2+3}\right)}$$

Example. $y=f(x)=\dfrac{(x+1)^{1/3}}{(1-3x)^4}$

method 1: Quotient rule: (left as an ex.)

method 2: (logarithmic dif.)

$\ln\left(\frac{a}{b}\right)=\ln a-\ln b$

S1) $\ln y=\ln\left[\dfrac{(x+1)^{1/3}}{(1-3x)^4}\right]$

$\ln y=\ln(x+1)^{1/3}-\ln(1-3x)^4$

$\boxed{\ln y=\frac{1}{3}\ln(x+1)-4\ln(1-3x)}$

$\ln(u)'=\frac{u'}{u}$

S2) $\dfrac{y'}{y}=\dfrac{1}{3}\cdot\dfrac{1}{x+1}-4\,\dfrac{-3}{1-3x}$

$y'=y\left(\dfrac{1}{3x+3}+\dfrac{12}{1-3x}\right)$

$y'=\dfrac{(x+1)^{1/3}}{(1-3x)^4}\left(\dfrac{1}{3x+3}+\dfrac{12}{1-3x}\right)$

$$f(x) = \frac{5}{1+e^{-x}} = 5\cdot(1+e^{-x})^{-1}$$

$$f'(x) = 5\cdot -1\cdot(1+e^{-x})^{-2}\cdot\frac{d}{dx}(1+e^{-x}) \quad \text{exp}(-x)$$

$$= -5\cdot(1+e^{-x})^{-2}\cdot(0+e^{-x}\cdot(-1))$$

$$= -5(1+e^{-x})^{-2}\cdot -e^{-x}$$

$$= \boxed{5(1+e^{-x})^{-2}\cdot e^{-x}} > 0$$

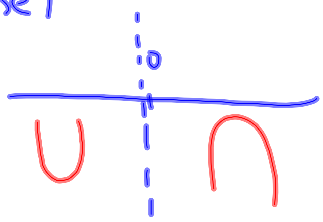
Caution: $\frac{d}{dx}(e^{-x}) = -1\cdot e^{-x}$ $\quad u = -x.$

$$\frac{d}{dx}(e^{u}) = u'\cdot e^{u}$$

Since $f' > 0 \Rightarrow f$ is increasing.

No relative max, and min.

$x = 0$ is an inflection point (left as an exercise)



$$\lim_{x\to+\infty}\frac{5}{1+e^{-x}} = 5$$

$$\lim_{x\to-\infty}\frac{5}{1+e^{-x}} = 0$$

$$\lim_{x\to+\infty} e^{-x} = \lim_{x\to+\infty}\frac{1}{e^{x}} = 0$$

$$\lim_{x\to-\infty}\frac{1}{e^{x}}$$

(x-axis)

$y = 0$ and $y = 5$ are horizontal asymptotes

Chapter 5

If $F' = f$, then F is called antiderivative of f.

The process of finding antiderivatives is called antidifferentiation or indefinite integral.

Example ① Find F such that $F' = 4x^3 = f$

$F = x^4$

$F = x^4 + 2$

$\therefore \boxed{F = x^4 + c}$

② Find F such that $F' = e^{2x} = f$.

$F = \frac{1}{2} e^{2x}$

Because $F' = \frac{1}{2} \cdot 2 \cdot e^{2x} = e^{2x}$

$$\boxed{F = \frac{1}{2} e^{2x} + C}$$

$$\frac{d}{dx}(e^u) = u' \cdot e^u$$

The Family of all antiderivatives of f is written by

$$\int f(x)\,dx = F(x) + C,$$

and called by indefinite integral of f.

Apr 19-10:32

$$\int 4x^3\,dx = x^4 + C$$

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

$$\int x\,dx = \frac{1}{2}\cdot x^2 + C$$

$$\int 1\cdot dx = x + C$$

$$\int 0\cdot dx = C$$

Apr 19-10:42

Integrals of some common Functions.

① $\int dx = \int 1 . dx = x + C$

② $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1.$

③ $\int \frac{1}{x} dx = \ln|x| + C$

④ $\int e^{kx} dx = \frac{1}{k} e^{kx} + C$

⑤ $\int k \cdot f(x) dx = k \cdot \int f(x) dx.$

Example. $\int (x^2 - 2\sqrt{x} + 1) dx.$

We break up the integral over sum and substraction:

$$\int (x^2 - 2\sqrt{x} + 1) dx = \int x^2 dx - \int 2 \underbrace{\sqrt{x}}_{= x^{1/2}} dx + \int 1 dx$$

$$= \frac{x^3}{3} + C_1 - 2 \cdot \frac{x^{3/2}}{3/2} + C_2 + x + C_3$$

$$= \frac{x^3}{3} + C_1 - \frac{4}{3} x^{3/2} + C_2 + x + C_3$$

$$= \boxed{\frac{x^3}{3} - \frac{4}{3} x^{3/2} + x + C,} \quad C = C_1 + C_2 + C_3$$

Example. $\int \left(x^{-2} - \frac{1}{x} + e^{2x} \right) dx = ?$

(Break up the integral over sum and substraction)

$$\int \left(x^{-2} - \frac{1}{x} + e^{2x} \right) dx = \int x^{-2} dx - \int \frac{1}{x} dx + \int e^{2x} dx$$

$$= \boxed{\frac{x^{-1}}{-1} - \ln|x| + \frac{1}{2} e^{2x} + C}$$

Example : $\int q^2 dq = \frac{q^3}{3}$ $\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$

marginal cost $= \frac{dC}{dq} = 3q^2 - 60q + 400$

$= C' = 3q^2 - 60q + 400$

$C(q) = \int (3q^2 - 60q + 400)\, dq$

$= 3\int q^2 dq - 60\int q\, dq + 400\int dq$

$= \cancel{3} \cdot \frac{q^3}{\cancel{3}} - \overset{30}{\cancel{60}} \cdot \frac{q^2}{\cancel{2}} + 400q + C$

$$\boxed{C(q) = q^3 - 30q^2 + 400q + C}$$

For $q = 2$, $C = 900$

$900 = 2^3 - 30 \cdot 2^2 + 400 \cdot 2 + C$

$\underset{100}{\cancel{900}} = 8 - 120 + \cancel{800} + C$

$C = 100 - 8 + 120 = 212$

$$\boxed{C(q) = q^3 - 30q^2 + 400q + 212}$$

$C(5) = 5^3 - 30 \cdot 5^2 + 400 \cdot 5 + 212 = \$1{,}587.$

Objective: Integral by substitution:

- Choose a substitution: $u = u(x)$
- $du = u'(x)\,dx$.
- Put du and u in the integral and calculate it by means of common integrals.

Example. Calculate $\int (3x+1)^2\,dx$ by expanding and by a substitution.

Expanding: $(a+b)^2 = a^2 + 2ab + b^2$, $a = 3x$, $b = 1$

$(3x+1)^2 = 9x^2 + 6x + 1$ $\qquad \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1}}$

$\int (9x^2+6x+1)\,dx = \int 9x^2\,dx + \int 6x\,dx + \int 1\,dx$

$= 9 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^2}{2} + x + C$

$= \boxed{3x^3 + 3x^2 + x + C}$

Substitution: $\int (3x+1)^2\,dx$.

Let S1) $u = 3x+1$ ✓

S2) $du = 3\,dx$ ✓ (derivative) $\qquad dx = \frac{du}{3}$

S3) $\int (3x+1)^2\,dx = \int u^2 \cdot \frac{du}{3} = \frac{1}{3}\int u^2\,du$

$= \frac{1}{3}\,\frac{u^3}{3} + C = \boxed{\frac{1}{3}\,\frac{(3x+1)^3}{3} + C}$

Ex: $\int e^{-x}\sqrt{e^{-x}+7}\,dx$

$(e^u)' = u' \cdot e^u$

S1) $u = e^{-x}+7$

S2) $du = -e^{-x}\,dx$

S3) $\int e^{-x}\sqrt{e^{-x}+7}\,dx = \int \cancel{e^{-x}}\sqrt{u}\cdot\frac{du}{-\cancel{e^{-x}}}$

$= -\int u^{1/2}\,du = -\frac{u^{3/2}}{3/2}+C$

$= -(e^{-x}+7)^{3/2}\cdot\frac{2}{3}+C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\frac{dA}{dt} = \cancel{A} = 40+0.026A$

$\int \frac{dA}{40+0.026A} = \int dt$

$= t + C_1$

$\int \frac{dA}{40+0.026A}$

S1) $u = 40+0.026A$

S2) $du = 0.026\,dA$

$= \int \frac{\frac{du}{0.026}}{u} = \frac{1}{0.026}\int\frac{du}{u}$

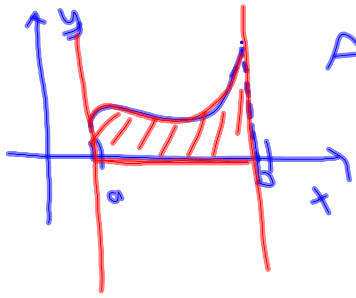
$= \frac{1}{0.026}\ln u + C = \frac{1}{0.026}\ln(40+0.026A) + C$

Apr 24-10:17

Objective: Definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where}$$

F is an antiderivative of f.



Area: $\int_a^b f(x)\,dx$.

example. $\int_{x=0=a}^{x=2=b} (\underbrace{4-x^2}_{f})\,dx = F(2) - F(0)$

$F(x) = 4x - \frac{1}{3}x^3 + C$

$\int_0^2 (4-x^2)\,dx = (4\cdot 2 - \frac{1}{3}2^3 + \cancel{C}) - (4\cdot 0 - \frac{1}{3}0^3 + \cancel{C})$

$= 8 - \frac{8}{3} = 16/3$

16/3 is the area between $4-x^2$, X axis, $x=0$, $x=2$.

$\int_{-2}^{1} \underline{x e^{-x^2}}\,dx$

S1) $u = x^2$

S2) $du = 2x\,dx$

$x\,dx = \frac{du}{2}$

$= \int_{-2} \frac{du}{2} \cdot e^{-u} = \frac{1}{2}\int_{-2}^{1} e^{-u}\,du$

$\int e^{-x}\,dx = -e^{-x} + C$

$= -\frac{1}{2} e^{-u} + C$ upper $x=1$

$= -\frac{1}{2} e^{-x^2} + C$ lower $x=-2$

$= (-\frac{1}{2} e^{-1^2} + C) - (-\frac{1}{2} e^{-4}) =$

$= -0.1$

Net Change: $\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\,dt$

$$P'(t) = 0.7t^4 - 7.7t^3 + 26.6t^2 - 28t$$

$$\int_3^6 P'(t)\,dt = P(6) - P(3)$$

$$\int_6^{10} 3(q-4)^2\,dq = 3\cdot\int_6^{10} \underbrace{(q-4)^2}_{q^2-8q+16}\,dq$$

S1. $u = q-4$

S2. $du = dq$

$$= 3\int_6^{10} u^2\,du = \cancel{3}\cdot\frac{u^3}{\cancel{3}} + c\,\Big|_{q=6}^{q=10}$$

$$= (q-4)^3 + c\,\Big|_{q=6}^{q=10}$$

$$= \left[(10-4)^3 + \cancel{c}\right] - \left[(6-4)^3 + \cancel{c}\right]$$

$$= 6^3 - 2^3 = 208.$$